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HIDE AND SEEK FROM A FIXED BASE

by

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An abstract hide and seek game is solved, the unique feature of which is that the hiders are constrained to return to a fixed point periodically.

HIDE AND SEEK FROM A FIXED BASE

by

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Background

Suppose that a force of "hiders" has an area A available within which to hide from a force of "seekers." Each side can distribute its forces arbitrarily within A. If the density of seeker effort at the location of any given hider is s, then the hider is assumed to escape with probability f(s), where f(s) is some decreasing, convex function of s. The hiders want to maximize the probability of escape, and the seekers want to minimize it. If no further constraints are imposed, it is not difficult to show that each side should allocate its forces uniformly throughout the region, and that the escape probability as the value of a two-person zero sum game is f(S/A), where S is the total amount of seeker effort. Our object in this report is to investigate the impact of constraints on the motion of the hiders. Specifically, we want to investigate what happens if the hiders are required to visit a particular point (a port, typically) on the boundary of A every t, while never travelling at a speed exceeding v.

We can anticipate that the escape probability will be $f(S/A)$ when the product vt is "large," and 0 when vt is "small."

In order to simplify the analysis, the following assumptions are made:

a) the region is a circular sector (wedge)

b) the revisit point is at the apex of the wedge

c) $f(s) = 1/(1 + s)$.

In Figure 1, the heaviness of the shading indicates the density of seeker effort for a typical seeker strategy. Note that the effort is dense in the vicinity of the apex, since the hiders must all transit through that area in order to visit the apex. A typical hider "tour" is shown. The hider basically picks a direction at random and a range from a distribution introduced in the next section, goes to the point and stays in the vicinity of that point until it is time to return to the apex. Each hider picks an independent tour after each visit to the apex.

Results

Let

θ = angle of the circular sector

r_{sec} = radius of the circular sector

$A = \theta r_{sec}^2/2$ = area of sector

v = hider speed

t = revisit time

$r_{max} = vt/2$ = maximum range of the hiders

S = total amount of seeker effort

$Y = S/(\theta\, r_{max}^2)$

$U = r_{sec}/r_{max}$

The value of the game (escape probability) depends on the two dimensionless quantities Y (a normalized amount of seeker effort) and U (a normalized sector size); call it $P(U,Y)$. Figure 2 shows $P(U,Y)$ as a function of Y for several values of U. Since $U = 1$ corresponds to the case where the maximum range of the hiders is equal to the sector radius, all values of $U > 1$ follow the same curve as for $U = 1$. The curves in Figure 2 are equivalent to:

(1) Let $V = 1 - \sqrt{1 - U^2}$

Case 1: for $Y \leq V^2/6$, $P(U,Y) = 1 - \sqrt{2Y/3}$

(2)

Case 2: for $Y \geq V^2/6$, $P(U,Y) = V/(Y + U^2/2)$

Standard limiting operations show that

$$\lim_{r_{max} \to 0} P(U,Y) = 0$$

(3)

$$\lim_{r_{max} \to \infty} P(U,Y) = 1/(1 + S/A) ,$$

as anticipated.

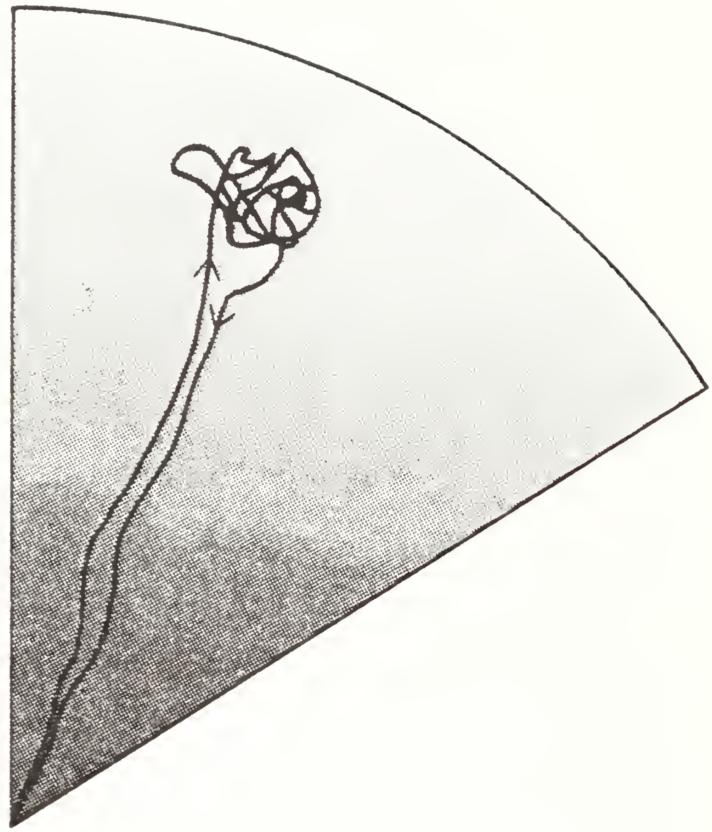

FIGURE 1

ILLUSTRATING A HIDER PATH AND A SHADED SEEKER DENSITY

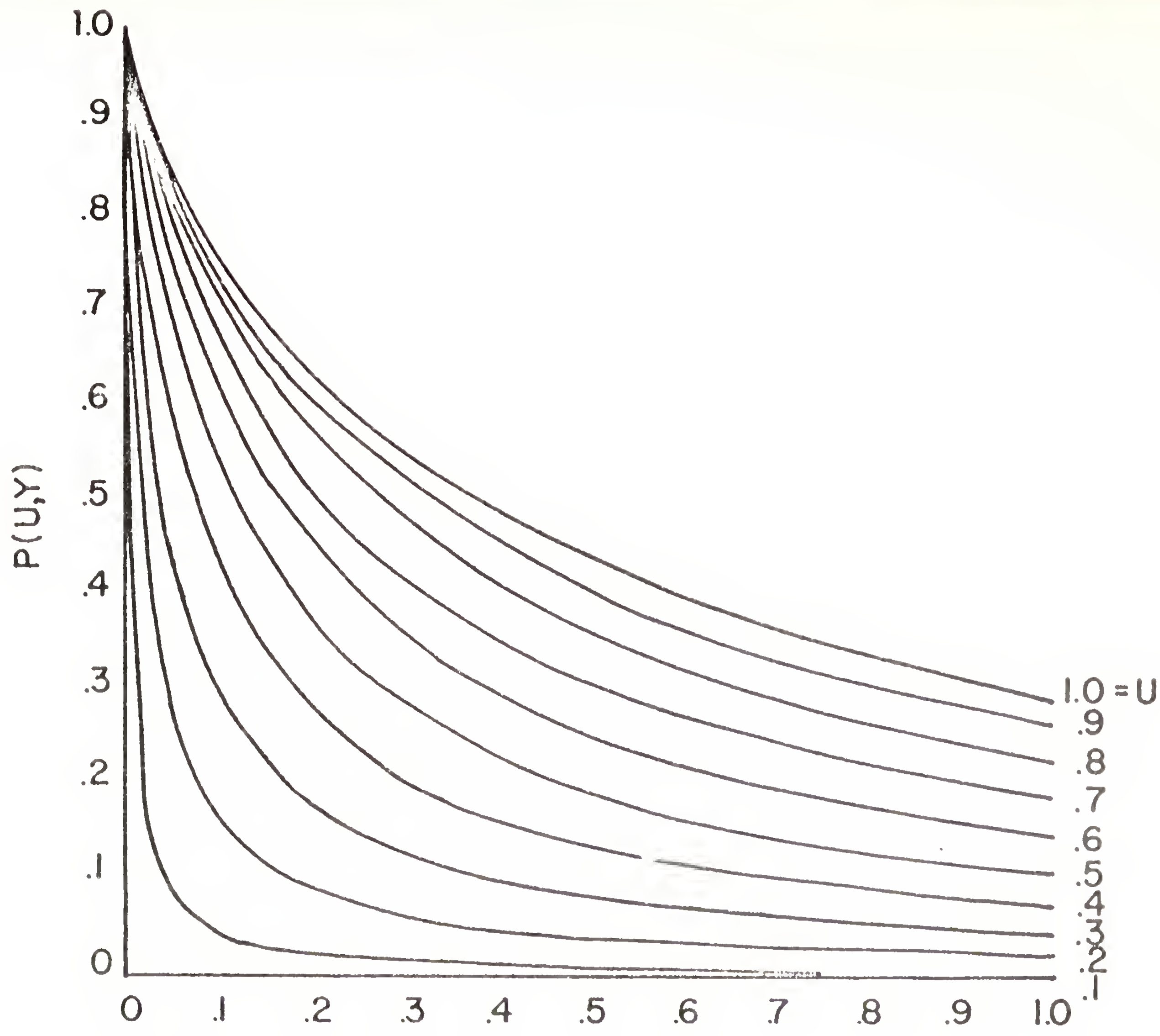

FIGURE 2. PROBABILITY OF SURVIVAL vs Y FOR VARIOUS U.

For example, suppose

r_{max} = 5000 mi

r_{sec} = 1000 mi

S = (25 hr holding time)(10^5 sq mi/hr search rate)

= 25×10^5 sq mi

θ = 1 radian

Then $Y = .1$, $U = .2$, and $V = .02$. This is Case 2, and $P(U,Y) = .164$. If r_{max} were "very large," we would have $P(U,Y) = 1/6$.

The optimal strategy for the hiders is to pick an angle at random and a range from the distribution $F(ur_{max})$, where

$$(4) \qquad F(u) = \begin{cases} 0 & \text{for } 0 \le u \le V \\ \dfrac{(u-V)^2}{2V(1-u)} & \text{for } V \le u \le U, \end{cases}$$

and where V is as earlier defined. Qualitatively, the hider have a tendency to pick large ranges, with Vr_{max} being the smallest range picked.

Let $y(u)$ be the density of seeker effort at range ur_{max}, let $T = \sqrt{6Y}$, and let $K = (Y + U^2/2)/[V(1 - V/3)]$. Then the optimal density $y(u)$ is

(5) Case 1. $$y(u) = \begin{cases} \sqrt{T/u} - 1 & \text{for } u \leq T \\ 0 & \text{for } u \geq T \end{cases}$$

(6) Case 2. $$y(u) = \begin{cases} K\sqrt{V/u} - 1 & \text{for } u \leq V \\ K - 1 & \text{for } u \geq V \end{cases}$$

Qualitatively, the searchers have a tendency to cluster near the apex, particularly in Case 1, $(Y \leq V^2/6)$. The density is actually unbounded near the apex; that is, $\lim_{u \to 0} y(u) = \infty$.

The proof that the functions given above represent the value of the game and the optimal strategies for the two sides is the subject of the next section.

Exact Statement of the Problem

Let u be range from the apex measured in units of r_{max}, so that the hiders must pick a range u for each tour in the interval $[0,U]$. Let $F(v)$ be the C.D.F. used by the hiders. Then the hiders spend $G(u)$ of their time within u of the apex, where

(7) $$G(u) = F(u) + u(1 - F(u)) .$$

Formula (7) is true because a hider will be within u of the apex throughout its patrol period if it picks a range smaller than or equal to u, and will spend a fraction u of its patrol time within u of the apex even if it picks a range greater than u. $F(u)$ can be any C.D.F. defined on $[0,U]$, but $G(u)$ cannot, which is what makes the problem non-trivial.

Since $y(u)$ is the density of seeker effort at range u, the escape probability for a hider averaged over time is

$$A(F,y) = \int_0^U (1 + y(u))^{-1} \, dG(u) \,, \tag{8}$$

where

$$\int_0^U y(u) \, u \, du = Y \quad \text{and} \quad y(u) \geq 0 \;. \tag{9}$$

Equations (7), (8), and (9) define a two-person zero sum game where the hiders select a C.D.F. $F(u)$ on $[0,U]$ and the seekers select $y(u)$ according to (9). We next show that the results quoted earlier constitute a saddle point of this game.

Proof of Results

The results shown below were discovered by using the theory of optimal control, but we will prove that the game has been solved by showing that the solution offered is a saddle point. While this is analytically simpler, it will not motivate the results.

We must show that

$$\max_F A(F,y^*) = P(U,Y) = \min_y A(F^*,y) ,$$

where F^* and y^* are the functions given earlier.

<u>Proof that</u> $P(U,Y) = \min_y A(F^*,y)$

Let $G^*(u) = F^*(u) + u(1 - F^*(u))$. Using (1) and (4), $F^*(U) = 1$, so also $G^*(U) = 1$. After substitution and simplification,

$$G^*(u) = \begin{cases} u & \text{for } u \leq V \\ \dfrac{u^2 + V^2}{2V} & \text{for } V \leq u \leq U \end{cases}$$

Let $g(u) = (d/du)G^*(u)$. Then we have

$$A(F^*,y) = \int_0^U g(u)/(1 + y(u))\, du ,$$

where

$$g(u) = \begin{cases} 1 & \text{for } u \leq V \\ u/V & \text{for } V \leq u \leq U \end{cases}$$

Consider the Lagrangian

$$h(y) = \int_0^U \left[\frac{g(u)}{1 + y(u)} + \lambda u\, y(u)\right] du ,$$

which is to be minimized subject to $y(u) \geq 0$. We minimize for each u separately by differentiation, obtaining the minimizing function $\tilde{y}$:

$$\tilde{y}(u) = \left(\sqrt{\frac{g(u)}{\lambda u}} - 1\right)^+ ,$$

where $+$ indicates that $\tilde{y}(u)$ is to be 0 rather than negative. If $\lambda V < 1$, $y(u) > 0$ for all u, and

$$(10) \qquad A(F^*,\tilde{y}) = \int_0^U \sqrt{\lambda u g(u)}\, du$$

$$= \sqrt{\lambda}\left[\int_0^V \sqrt{u}\, du + \int_V^U u/\sqrt{V}\, du\right]$$

$$= \sqrt{\lambda/V}\left[\frac{2}{3} V^2 + (U^2 - V^2)/2\right]$$

$$= \sqrt{\lambda/V}\ [V^2/6 + U^2/2]$$

If $\tilde{y}$ is to be feasible, we must also have

$$(11) \qquad Y = \int_0^U u\tilde{y}(u)\, du = \int_0^V \sqrt{u/\lambda}\, du + \int_V \frac{u}{\sqrt{\lambda V}}\, du - U^2/2$$

$$= \frac{1}{\sqrt{\lambda V}}\ [V^2/6 + U^2/2] - U^2/2$$

Since $U^2 + V^2 = 2V$, $V^2/6 + U^2/2 = V(1 - V/3)$. Solving (11) for $\sqrt{\lambda}$ and then substituting $\sqrt{\lambda}$ in (10), we obtain $A(F^*,\tilde{y}) = P(U,Y)$, and also $\lambda V < 1$ if and only if $Y > V^2/6$.

If $\lambda V \geq 1$, $\tilde{y}(u) = 0$ for $u \geq 1/\lambda$. Let $T = 1/\lambda$. Then

$$A(F^*,\tilde{y}) = \int_0^T \sqrt{\lambda u}\, du + 1 - T = 1 - T/3 , \tag{12}$$

and if $\tilde{y}$ is to be feasible we must have

$$Y = \int_0^T u(\sqrt{1/\lambda u} - 1)\, du = 2T^2/3 - T^2/2 = T^2/6 \tag{13}$$

Solving (13) for T and substituting in (12), we obtain $A(F^*,\tilde{y}) = P(U,Y)$, and also $\lambda V \geq 1$ if and only if $Y \leq V^2/6$.

According to Everett's theorem [1] on Lagrange multipliers, $A(F^*,y) \geq A(F^*,\tilde{y})$, so we have shown that $P(U,Y) = \min_y A(F^*,y)$. We also note that $\tilde{y} = y^*$.

<u>Proof that</u> $P(U,Y) = \max_F A(F,y^*)$

Since $y^*(u)$ is differentiable, we can integrate $A(F,y^*)$ by parts to obtain

$$A(F,y^*) = \frac{G(U)}{1 + y^*(U)} - \int_0^U G(u)\, B(u)\, du ,$$

where

$$B(u) = -\left(\frac{d}{du}\, y^*(u)\right)/(1 + y^*(u))^2 .$$

In both Cases I and II, $B(u) \geq 0$ for $u \leq V$, and $B(u) = 0$ for $u \geq V$ (note $T \leq V$ in (5)). Since $G(u) = u + F(u)(1-u) \geq u$,

$$A(F,y^*) \leq \frac{1}{1 + y^*(U)} - \int_0^U uB(u)\,du \ .$$

But it is also true that $G^*(u) = u$ for $u \leq V$, so $A(F,y^*) \leq A(F^*,y^*)$ for any F. But we already know $A(F^*,y^*) = P(U,Y)$, so the proof is complete.

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REFERENCE

(1) Everett, H. III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources," Opns. Res. 11, 399-417 (1963).

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